Keep your homework!

Warm-up: Find and classify the critical points of $g(x,y) = e^{3x} + y^3 - 3ye^x$.
What do you think it looks like?

$$\nabla g = (0,0) = (3e^{3x} - 3ye^x,\ 3y^2 - 3e^x)$$

$$\left.\begin{array}{l} 3e^{3x} = 3ye^x \Rightarrow \\ 3y^2 = 3e^x \Rightarrow \end{array}\right. \left.\begin{array}{l} e^{3x} = ye^x \\ y^2 = e^x \end{array}\right\}$$

$$y = \frac{e^{3x}}{e^x} = e^{2x} \Rightarrow y^2 = e^{4x} = e^x$$

$$\Rightarrow e^{3x} = 1 \Rightarrow x = 0$$

$$y = 1$$

Only critical pt is $(x,y) = (0,1)$

Hessian:

$$g_{xx} = 9e^{3x} - 3ye^x = 9 - 3 = 6$$

$$g_{xy} = -3e^x = -3$$

$$g_{yy} = 6y = 6$$

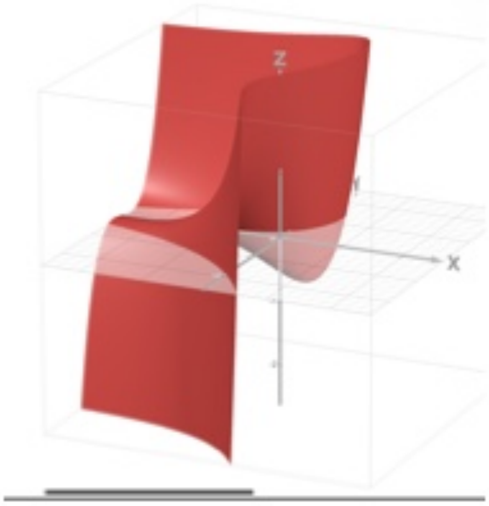
$$H = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} \xrightarrow{\text{eigenv's}} \begin{vmatrix} 6-\lambda & -3 \\ -3 & 6-\lambda \end{vmatrix}$$

$$= (6-\lambda)^2 - 9 \Rightarrow \lambda^2 - 12\lambda + 36 - 9 = 0$$

$$\lambda^2 - 12\lambda + 27 = 0$$

$$(\lambda - 3)(\lambda - 9) = 0 \qquad \lambda = 3, 9.$$

$\Rightarrow$ $(0,1)$ is the only critical pt, and it's a local min.



Note — it's not a global min !!

Moral: If we have only one critical pt & it's a local min, in ≥ 2 dimensions, that's not enough information to conclude it's a global min.

2.39b Find & classify the cr. pts of

$$f(x,y) = (x^2+x+1)\, e^{-\frac{x^2}{10} - y^2}.$$

$$\nabla f = \left((2x+1)e^{\cdots} + (x^2+x+1)\left(-\frac{x}{5}\right)e^{\cdots},\ (x^2+x+1)(-2y)\, e^{\cdots} \right)$$

$$= (0,0)$$

$e^{\cdots} > 0 \Leftarrow$ divide by this.

$$\left(-\frac{x^3}{5} - \frac{x^2}{5} + \frac{9x}{5} + 1,\ -2y(x^2+x+1) \right) = (0,0)$$

$$-x^3 - x^2 + 9x + 5 = 0$$

$$-2y\underbrace{(x^2+x+1)}_{>0} = 0 \implies y = 0$$

$x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} > 0.$

```
f(x,y)=(x^2+x+1)*e^(-x^2/10-y^2)
fx = diff(f(x,y),x)
fy = diff(f(x,y),y)
show(factor(fx))
show(factor(fy))
eq1 = fx==0
eq2 = fy==0
solns = solve([eq1,eq2],x,y,solution_dict=True)
#show(solns)
ans=[[s[x].n(),s[y].n(),f(s[x],s[y]).n()] for s in solns]
show(ans)
fxx = diff(fx,x)
fxy = diff(fx,y)
fyy = diff(fy,y)
xx=-3.279
yy=0
h11=fxx.subs(x=xx,y=yy)
h12=fxy.subs(x=xx,y=yy)
h22=fyy.subs(x=xx,y=yy)
H=matrix([[h11,h12],[h12,h22]])
show(H)
show(H.eigenvalues())
```

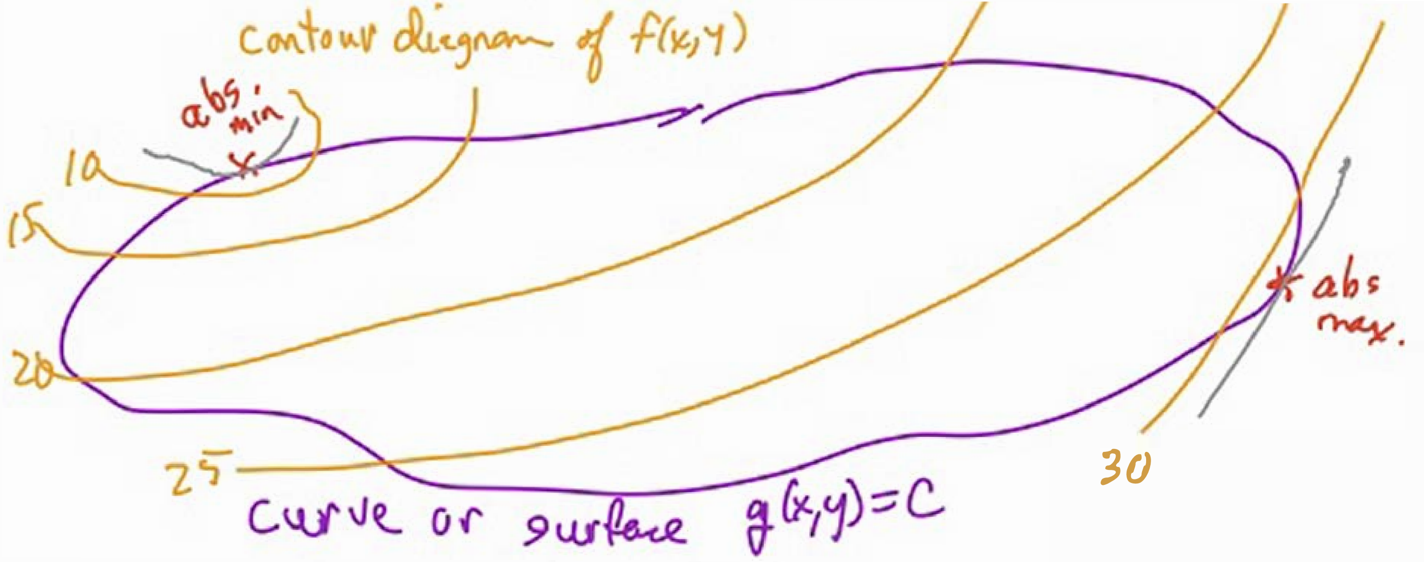
different cr. pts.

Sagemath code to do calculations

Solving for critical points (including max's & mins) of a function $f(x,y,\ldots)$ restricted to a curve/surface $g(x,y,\ldots)=C$

constant.

Example: Find the maximum & minimum values of $f(x,y)=3x-5y+10$ where restricted to the ellipse $\underbrace{\frac{x^2}{4}+\frac{y^2}{9}}_{g(x,y)}=\underbrace{1}_{C}$.



Moral: To find cr. pts of $f(x,y,\dots)$, we need to find pts where the level sets of f are tangent to $g(x,y)=C$.

$\Longleftrightarrow$ find (x,y) where $\nabla f \parallel \nabla g$

$\Longleftrightarrow$ $\nabla f = \lambda \nabla g$ for some constant λ

Recipe: To find max's/min's/saddle pts of $f(x,y,\dots)$ restricted to $g(x,y,\dots) = C$,

we solve this system of equations: for (x,y,λ)

$$g(x,y)=C$$
$$f_x = \lambda g_x$$
$$f_y = \lambda g_y$$
$$\vdots$$

don't care what λ is.

This is the Lagrange multipliers method

Example

Example: Find the maximum & minimum values of $f(x,y) = 3x - 5y + 10$ where restricted to the ellipse $\underbrace{\frac{x^2}{4} + \frac{y^2}{9}}_{g(x,y)} = \underbrace{1}_{c}$.

$\nabla f = \lambda \nabla g \qquad g = c$

equations: $\frac{x^2}{4} + \frac{y^2}{9} = 1$

$f_x = 3 = \lambda \cdot \frac{x}{2} = \lambda g_x$

$f_y = -5 = \lambda \frac{2y}{9}$

$\Rightarrow \lambda = \frac{6}{x} = \frac{-45}{2y}$

$\Rightarrow 12y = -45x$

$y = \frac{-45x}{12} = \frac{-15x}{4}$

plug back into

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

solve for x

Finishing question:

$$\frac{x^2}{4} + \frac{\left(\frac{-15x}{4}\right)^2}{9} = 1$$

$$\left[\frac{1}{4} + \frac{\overset{25}{\cancel{225}}}{\cancel{9}\cdot 16}\right]x^2 = 1$$

$$\Rightarrow \frac{29}{16}x^2 = 1$$

$$\Rightarrow x^2 = \frac{16}{29} \Rightarrow x = \pm\frac{4}{\sqrt{29}}$$

$$y = \frac{-15x}{4} = \mp\frac{15}{\sqrt{29}}$$

Solutions (cr pts) are $(x,y) = \left(\frac{4}{\sqrt{29}}, \frac{-15}{\sqrt{29}}\right)$

$(x,y) = \left(\frac{-4}{\sqrt{29}}, \frac{15}{\sqrt{29}}\right)$

$f(\cdot,\cdot) = \frac{12}{\sqrt{29}} + \frac{75}{\sqrt{29}} + 10 = \frac{87}{\sqrt{29}} + 10$

$f(\cdot,\cdot) = \frac{-12}{\sqrt{29}} - \frac{75}{\sqrt{29}} + 10 = \frac{-87}{\sqrt{29}} + 10$

$f(x,y) = 3x - 5y + 10$

Max

Min:

$\therefore \left(\frac{4}{\sqrt{29}}, \frac{-15}{\sqrt{29}}\right)$ is global max, max value $= \frac{87}{\sqrt{29}} + 10$

$\left(-\frac{4}{\sqrt{29}}, \frac{15}{\sqrt{29}}\right)$ is global min, min value $= -\frac{87}{\sqrt{29}} + 10$.